

OPERATOR VALUED MAPS ON HILBERT C^* -MODULES

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ABSTRACT. We provide a characterization for operator valued completely bounded linear maps on Hilbert C^* -modules in terms of φ -maps. Also, we show that for every operator valued completely positive map φ on a C^* -algebra \mathcal{A} , there is a unique (up to multiplication by a unitary operator) non-degenerate φ -map on each Hilbert \mathcal{A} -module.

1. INTRODUCTION

The study of φ -maps on Hilbert C^* -modules has increased significantly during recent decades. In this context, several concepts such as representation theory of Hilbert C^* -modules, dilation theory of φ -maps and CP-extendable maps were studied ([1, 3, 4, 5, 6, 9, 11, 12, 16, 17]). Therefore it becomes natural to concentrate on φ -maps as important maps on Hilbert C^* -modules. To confirm this statement, we show that an operator valued map on a Hilbert C^* -module is completely bounded if and only if it can be decomposed to a bounded operator and a φ -map, for a completely positive map φ on the underlying C^* -algebra of the Hilbert C^* -module.

Moreover, for a given operator valued completely positive map φ on a C^* -algebra \mathcal{A} and its minimal Stinespring dilation π , we construct a φ -map and a π -representation for each Hilbert \mathcal{A} -module \mathcal{E} and also we show that every non-degenerate φ -map (non-degenerate π -representation) on \mathcal{E} is a unitary operator multiple of the above constructed φ -map (π -representation).

We denote Hilbert spaces by $\mathcal{H}, \mathcal{K}, \mathcal{L}$. The set of all bounded operators between Hilbert spaces \mathcal{H}, \mathcal{K} is denoted by $\mathcal{B}(\mathcal{H}, \mathcal{K})$, and $\mathcal{B}(\mathcal{H}) := \mathcal{B}(\mathcal{H}, \mathcal{H})$.

Assume \mathcal{E} is a Hilbert C^* -module over a unital C^* -algebra \mathcal{A} . The linking C^* -algebra of \mathcal{E} is denoted by $\mathcal{L}(\mathcal{E})$ and defined as $\mathcal{L}(\mathcal{E}) := \left\{ \begin{bmatrix} T & x \\ y^* & a \end{bmatrix} : a \in \mathcal{A}, T \in \mathbb{K}(\mathcal{E}), x, y \in \mathcal{E} \right\}$, where $\mathbb{K}(\mathcal{E})$ is the set of compact operators on \mathcal{E} . Also, for arbitrary given maps $\rho : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$, $\sigma : \mathbb{K}(\mathcal{E}) \rightarrow \mathcal{B}(\mathcal{K})$ and $\Psi : \mathcal{E} \rightarrow \mathcal{B}(\mathcal{H}, \mathcal{K})$, the map $\begin{bmatrix} T & x \\ y^* & a \end{bmatrix} \mapsto \begin{bmatrix} \sigma(T) & \Psi(x) \\ \Psi(y)^* & \rho(a) \end{bmatrix}$ from $\mathcal{L}(\mathcal{E})$ into $\mathcal{B}(\mathcal{K} \oplus \mathcal{H})$ is denoted by $\begin{bmatrix} \sigma & \Psi \\ \Psi^* & \rho \end{bmatrix}$.

If $\varphi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ is a completely positive map and $\Phi : \mathcal{E} \rightarrow \mathcal{B}(\mathcal{H}, \mathcal{K})$ a map, then we say

- (1) Φ is *non-degenerate*, if $[\Phi(\mathcal{E})\mathcal{H}] = \mathcal{K}$.
- (2) Φ is a *φ -map*, if $\Phi(x)^*\Phi(y) = \varphi(\langle x, y \rangle)$, for all $x, y \in \mathcal{E}$.
- (3) Φ is a *representation* (or *ρ -representation*), if there is a $*$ -representation $\rho : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ such that Φ is a ρ -map.
- (4) Φ is a *completely semi- φ -map*, if $\Phi_n(x)^*\Phi_n(x) \leq \varphi_n(\langle x, x \rangle)$ for every $n \in \mathbb{N}$ and $x \in \mathbb{M}_n(\mathcal{E})$.

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(5) Φ is a *CP-extendable map*, if there exist completely positive maps $\phi_1 : \mathbb{K}(\mathcal{E}) \rightarrow \mathcal{B}(\mathcal{K})$ and $\phi_2 : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ such that $\begin{bmatrix} \phi_1 & \Phi \\ \Phi^* & \phi_2 \end{bmatrix} : \mathcal{L}(\mathcal{E}) \rightarrow \mathcal{B}(\mathcal{K} \oplus \mathcal{H})$, is a completely positive map.

(6) Φ is *dilatable* if there is a representation $\Psi : \mathcal{E} \rightarrow \mathcal{B}(\mathcal{H}', \mathcal{K}')$ and bounded operators $V : \mathcal{H} \rightarrow \mathcal{H}'$ and $W : \mathcal{K} \rightarrow \mathcal{K}'$ such that $\Phi(x) = W^* \Psi(x) V$, for every $x \in \mathcal{E}$.

Positive definite kernels are a non-linear version of completely positive maps which are older than their linear counterpart (see [2, 7, 8, 14]) for more details). A positive definite kernel on a set X is a two variables function $\phi : X \times X \rightarrow \mathcal{B}(\mathcal{H})$, where \mathcal{H} is a Hilbert space, such that for every choice of n elements in X such as $\{x_i\}_{i=1}^n$, $[\phi(x_i, x_j)] \in \mathbb{M}_n(\mathcal{B}(\mathcal{H}))_+$. From now on we use PD kernel to abbreviate positive definite kernel.

For a given PD kernel $\phi : X \times X \rightarrow \mathcal{B}(\mathcal{H})$ there is a standard way to construct another Hilbert space \mathcal{K} such that ϕ is decomposed into more tractable functions from X into $\mathcal{B}(\mathcal{H}, \mathcal{K})$ [7, 8].

Definition 1.1. Let X be a non-empty set and $\phi : X \times X \rightarrow \mathcal{B}(\mathcal{H})$ be a PD kernel. A Kolmogorov decomposition pair for ϕ is a pair (ν, \mathcal{K}) consists of a Hilbert space \mathcal{K} a map $\nu : X \rightarrow \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $\phi(x, y) = \nu(x)^* \nu(y)$. A Kolmogorov decomposition pair is called minimal when $[\nu(X)\mathcal{H}] = \mathcal{K}$.

The existence of the Kolmogorov decomposition pair for a PD kernel is a well-known result:

Theorem 1.2. Let $\phi : X \times X \rightarrow \mathcal{B}(\mathcal{H})$ be a PD kernel, then there is a Hilbert space \mathcal{K} and a map $\nu : X \rightarrow \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $\phi(x, y) = \nu(x)^* \nu(y)$, for all $x, y \in X$.

Remark 1.3. Minimal Kolmogorov decomposition pairs of ϕ are unique up to unitary equivalence. That is, if (ν, \mathcal{K}) is a minimal Kolmogorov decomposition of ϕ and (ν, \mathcal{L}) is an arbitrary Kolmogorov decomposition pair of ϕ , then there is a unique isometry $V : \mathcal{K} \rightarrow \mathcal{L}$ such that $V\nu(x) = \nu(x)$.

To every map $\Phi : X \rightarrow \mathcal{B}(\mathcal{H}, \mathcal{K})$, one can associate a PD kernel $\Lambda_\Phi : X \times X \rightarrow \mathcal{B}(\mathcal{H})$ by $\Lambda_\Phi(x, y) = \Phi(x)^* \Phi(y)$, which has Φ as its Kolmogorov decomposition. Also, a completely positive map $\varphi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ induces a PD kernel $\tilde{\varphi} : \mathcal{E} \times \mathcal{E} \rightarrow \mathcal{B}(\mathcal{H})$ by $\tilde{\varphi}(x, y) = \varphi(\langle x, y \rangle)$, on every Hilbert \mathcal{A} -module \mathcal{E} .

2. Main Theorems

The following theorem says that each operator valued completely bounded map on a Hilbert C^* -module is an operator multiple of some φ -map. We mention that a similar discussion can be found in [17, Section 3]. In fact, the main idea of the proof is to use the fact that the space $B(K \oplus H)$ is injective in the category of operator systems.

Theorem 2.1. Let \mathcal{E} be a right Hilbert C^* -module over a unital C^* -algebra \mathcal{A} and $\Phi : \mathcal{E} \rightarrow \mathcal{B}(\mathcal{H}, \mathcal{K})$ be a map. The following statements are equivalent

- (i) Φ is a completely bounded linear map.
- (ii) There is a completely positive map $\varphi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ and a φ -map $\Gamma : \mathcal{E} \rightarrow \mathcal{B}(\mathcal{H}, \mathcal{L})$ and a bounded operator $S : \mathcal{L} \rightarrow \mathcal{K}$ such that $\Phi(x) = S\Gamma(x)$, for all $x \in \mathcal{E}$.

The following lemma provides a representation theorem for completely positive maps on C^* -algebras, in term of maps on Hilbert C^* -modules.

Lemma 2.2. Let \mathcal{E} be a right Hilbert C^* -module over a unital C^* -algebra \mathcal{A} , $\varphi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ a completely positive map and (π, \mathcal{K}, V) be the minimal Stinespring dilation triple of φ . Then, there exists a triple $((\Phi_\varphi, \mathcal{H}_\varphi), (\Psi_\pi, \mathcal{K}_\pi), W_\varphi)$ consists of Hilbert spaces \mathcal{H}_φ and \mathcal{K}_π , a unitary operator $W_\varphi : \mathcal{H}_\varphi \rightarrow \mathcal{K}_\pi$, a non-degenerate φ -map $\Phi_\varphi : \mathcal{E} \rightarrow \mathcal{B}(\mathcal{H}, \mathcal{H}_\varphi)$ and a non-degenerate π -representation $\Psi_\pi : \mathcal{E} \rightarrow \mathcal{B}(\mathcal{K}, \mathcal{K}_\pi)$ such that $\Phi_\varphi(\cdot) = W_\varphi^* \Psi_\pi(\cdot) V$.

Now, we summarize some results about φ -maps on Hilbert C^* -modules. In fact, in the following theorem, the part (i) is the same as Bhat-Ramesh-Sumesh's theorem [5, Theorem 2.1] and also says that for every completely positive map on a C^* -algebra \mathcal{A} , such as $\varphi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$, there

is a unique (up to multiplication by a unitary operator) non-degenerate φ -map on each Hilbert \mathcal{A} -module. The part (ii) strengthen [4, Theorem 3.4] and characterizes completely semi- φ -maps as operator multiple of φ -maps. Also, the part (iii) is a similar result to (i) and finally, the part (iv) exposes the relation between every pair of φ -maps and π -representations on a same Hilbert C^* -module.

Theorem 2.3. *With the notations of the above lemma, one has*

(i) *a map $\Phi : \mathcal{E} \rightarrow \mathcal{B}(\mathcal{H}, \mathcal{H}')$ is a (non-degenerate) φ -map if and only if there exist a (unitary) isometry $S_\Phi : \mathcal{H}_\varphi \rightarrow \mathcal{H}'$, and a (unitary) coisometry $W : \mathcal{H}' \rightarrow \mathcal{K}_\pi$ such that $S_\Phi \Phi_\varphi = \Phi$ and $\Phi(\cdot) = W^* \Psi_\pi(\cdot) V$;*

(ii) *a map $\Phi : \mathcal{E} \rightarrow \mathcal{B}(\mathcal{H}, \mathcal{H}')$ is a (non-degenerate) completely semi- φ -map if and only if there exist a (dense range) contraction $S : \mathcal{H}_\varphi \rightarrow \mathcal{H}'$, and a (injective) contraction $W : \mathcal{H}' \rightarrow \mathcal{K}_\pi$ such that $S \Phi_\varphi = \Phi$ and $\Phi(\cdot) = W^* \Psi_\pi(\cdot) V$;*

(iii) *a map $\Psi : \mathcal{E} \rightarrow \mathcal{B}(\mathcal{K}, \mathcal{K}')$ is a (non-degenerate) π -representation if and only if there exists an (unitary) isometry $S_\Psi : \mathcal{K}_\pi \rightarrow \mathcal{K}'$ such that $\Psi(\cdot) = S_\Psi \Psi_\pi(\cdot)$;*

(iv) *if $\Psi : \mathcal{E} \rightarrow \mathcal{B}(\mathcal{K}, \mathcal{K}')$ is a π -representation and $\Phi : \mathcal{E} \rightarrow \mathcal{B}(\mathcal{H}, \mathcal{H}')$ is a φ -map, then there exists a partial isometry $W : \mathcal{H}' \rightarrow \mathcal{K}'$ such that $\Phi(\cdot) = W^* \Psi(\cdot) V$. Moreover W is unitary when Φ and Ψ are non-degenerate.*

Completely semi- φ -maps introduced in [4] as generalizations of φ -maps. By the above theorem, every completely semi- φ -map can be dilated to a representation of the Hilbert C^* -module and therefore it is a linear map. Also, CP-extendable maps introduced in [17] and the authors in [4, Theorem 4.2] showed that each operator valued map on a Hilbert C^* -module is dilatable if and only if it is CP-extendable. Therefore, we have the following result.

Corollary 2.4. *Let \mathcal{E} be a right Hilbert C^* -module over a unital C^* -algebra \mathcal{A} and $\Phi : \mathcal{E} \rightarrow \mathcal{B}(\mathcal{H}, \mathcal{K})$ be a map. The following statements are equivalent*

- (i) Φ is a completely bounded linear map.
- (ii) There is a completely positive map $\varphi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ and a φ -map $\Gamma : \mathcal{E} \rightarrow \mathcal{B}(\mathcal{H}, \mathcal{L})$ and a bounded operator $S : \mathcal{L} \rightarrow \mathcal{K}$ such that $\Phi(x) = S \Gamma(x)$, for all $x \in \mathcal{E}$.
- (iii) There is a completely positive map $\psi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ such that Φ is a completely semi- ψ -map.
- (iv) Φ is dilatable.
- (v) Φ is CP-extendable.

The following corollary is a well known theorem on completely bounded maps on C^* -algebras [15]. However, we can conclude it as a special case of the above result, since each C^* -algebra is a right Hilbert C^* -module over itself and also $\mathbb{K}(\mathcal{A}) \cong \mathcal{A}$, $\mathbb{M}_2(\mathcal{A}) \cong \mathcal{L}(\mathcal{A})$.

Corollary 2.5. *Let \mathcal{A} be a unital C^* -algebra. If $\psi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ is a completely bounded map, there exist completely positive maps $\phi_i : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$, $i = 1, 2$, such that the map $\begin{bmatrix} \phi_1 & \psi \\ \psi^* & \phi_2 \end{bmatrix} : \mathbb{M}_2(\mathcal{A}) \rightarrow \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ is completely positive.*

3. Proofs

Proof of Theorem 2.1:

(i) \Rightarrow (ii) : Assume Φ is completely bounded. Since $\mathcal{B}(\mathcal{K} \oplus \mathcal{H}) = \begin{bmatrix} \mathcal{B}(\mathcal{K}) & \mathcal{B}(\mathcal{H}, \mathcal{K}) \\ \mathcal{B}(\mathcal{K}, \mathcal{H}) & \mathcal{B}(\mathcal{H}) \end{bmatrix}$, we can consider Φ as a map from \mathcal{E} into $\mathcal{B}(\mathcal{K} \oplus \mathcal{H})$.

Let $\mathcal{L}_1(\mathcal{E}) := \left\{ \begin{bmatrix} T & x \\ y^* & a \end{bmatrix} : a \in \mathcal{A}, T \in \mathbb{K}_1(\mathcal{E}) := \mathbb{K}(\mathcal{E}) + \mathbb{C}I_{\mathcal{E}}, x, y \in \mathcal{E} \right\}$, be the unitization of the linking C^* -algebra of \mathcal{E} , then φ can be extended to a completely bounded map $\Psi : \mathcal{L}_1(\mathcal{E}) \rightarrow \mathcal{B}(\mathcal{K} \oplus \mathcal{H})$ by Wittstock's extension theorem. Then there is a $*$ -representation $\pi : \mathcal{L}_1(\mathcal{E}) \rightarrow \mathcal{B}(\mathcal{L})$ and bounded operators $V_i : \mathcal{K} \oplus \mathcal{H} \rightarrow \mathcal{L}$, $i = 1, 2$ such that $\Psi(X) = V_1^* \pi(X) V_2$ for every $X \in \mathcal{L}_1(\mathcal{E})$.

Using [1, Proposition 3.1] \mathcal{L} decomposes to $\mathcal{L}_2 \oplus \mathcal{L}_1$ for two orthogonal closed subspaces \mathcal{L}_1 and \mathcal{L}_2 and there exist $*$ -representations $\rho : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{L}_1)$, $\sigma : \mathbb{K}_1(\mathcal{E}) \rightarrow \mathcal{B}(\mathcal{L}_2)$ and a σ - ρ -representation $\Gamma_0 : \mathcal{E} \rightarrow \mathcal{B}(\mathcal{L}_1, \mathcal{L}_2)$ such that $\pi = \begin{bmatrix} \sigma & \Gamma_0 \\ \Gamma_0^* & \rho \end{bmatrix} : \mathcal{L}_1(\mathcal{E}) \rightarrow \mathcal{B}(\mathcal{L}_2 \oplus \mathcal{L}_1)$.

Since Ψ is an extension of Φ , and the operators V_i , $i = 1, 2$ has the matrix decompositions $V_i = \begin{bmatrix} S_{i,1} & S_{i,2} \\ S_{i,3} & S_{i,4} \end{bmatrix} \in \mathcal{B}(\mathcal{K} \oplus \mathcal{H}, \mathcal{L}_2 \oplus \mathcal{L}_1)$, $i = 1, 2$, one has

$$\begin{bmatrix} 0 & \Phi(x) \\ 0 & 0 \end{bmatrix} = \Psi \left(\begin{bmatrix} 0 & e \\ 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} S_{1,1} & S_{1,2} \\ S_{1,3} & S_{1,4} \end{bmatrix}^* \begin{bmatrix} \sigma(0) & \Gamma_0(x) \\ \Gamma_0(0)^* & \rho(0) \end{bmatrix} \begin{bmatrix} S_{2,1} & S_{2,2} \\ S_{2,3} & S_{2,4} \end{bmatrix},$$

for every $x \in \mathcal{E}$. Thus $\Phi(x) = S_{1,1}^* \Gamma_0(x) S_{2,4}$ for every $x \in \mathcal{E}$.

Now, if we set $\varphi(\cdot) = S_{2,4}^* \rho(\cdot) S_{2,4}$ and $\Gamma(\cdot) = \Gamma_0(\cdot) S_{2,4}$, then φ is a completely positive map, Γ is a φ -map and $\Phi(\cdot) = S_{1,1}^* \Gamma(\cdot)$.

(ii) \Rightarrow (i) : Let $\Phi(\cdot) = S\Gamma(\cdot)$. Let $[x_{ij}] \in M_n(\mathcal{E})$, then

$$\begin{aligned} \Phi_n([x_{ij}])^* \Phi_n([x_{ij}]) &= [\Gamma(x_{ji})^* S^*] [S\Gamma(x_{ij})] = [\Gamma(x_{ji})^*] \text{diag}(S^*, \dots, S^*) \text{diag}(S, \dots, S) [\Gamma(x_{ij})] \\ &\leq \|S\|^2 [\Gamma(x_{ji})^*] [\Gamma(x_{ij})] = \|S\|^2 \varphi_n(\langle [x_{ij}], [x_{ij}] \rangle) \end{aligned}$$

Therefore,

$$\|\Phi_n([x_{ij}])\|^2 = \|\Phi_n([x_{ij}])^* \Phi_n([x_{ij}])\| \leq \|S\|^2 \|\varphi\|_{cb} \|[x_{ij}]\|^2$$

and then Φ is a completely bounded map. □

Proof of Lemma 2.2:

Define $\tilde{\varphi} : \mathcal{E} \times \mathcal{E} \rightarrow \mathcal{B}(\mathcal{H})$ by

$$\tilde{\varphi}(x, y) := \varphi(\langle x, y \rangle_{\mathcal{A}})$$

for all $x, y \in \mathcal{E}$. Note that the \mathcal{A} -valued inner-product on \mathcal{E} , $\langle \cdot, \cdot \rangle_{\mathcal{A}} : \mathcal{E} \times \mathcal{E} \rightarrow \mathcal{A}$ is a PD kernel and φ is a completely positive map on \mathcal{A} , therefore $\tilde{\varphi}$ is a PD kernel on \mathcal{E} . There is a (unique) minimal Kolmogorov decomposition $(\Phi_\varphi, \mathcal{H}_\varphi)$ for $\tilde{\varphi}$, consists of a Hilbert space \mathcal{H}_φ and a map $\Phi_\varphi : \mathcal{E} \rightarrow \mathcal{B}(\mathcal{H}, \mathcal{H}_\varphi)$ such that the linear span of $\Phi_\varphi(\mathcal{E})\mathcal{H}$ is a dense subspace of \mathcal{H}_φ and $\tilde{\varphi}(x, y) = \Phi_\varphi(x)^* \Phi_\varphi(y)$ for all $x, y \in \mathcal{E}$. Thus Φ_φ is a non-degenerate φ -map from \mathcal{E} into $\mathcal{B}(\mathcal{H}, \mathcal{H}_\varphi)$.

Similarly, define $\tilde{\pi} : \mathcal{E} \times \mathcal{E} \rightarrow \mathcal{B}(\mathcal{K})$ by $\tilde{\pi}(x, y) := \pi(\langle x, y \rangle_{\mathcal{A}})$, for all $x, y \in \mathcal{E}$. A similar argument like above lines implies the existence of a (unique) minimal Kolmogorov decomposition pair $(\Psi_\pi, \mathcal{K}_\pi)$ for $\tilde{\pi}$, consists of a Hilbert space \mathcal{K}_π and a map $\Psi_\pi : \mathcal{E} \rightarrow \mathcal{B}(\mathcal{H}, \mathcal{K}_\pi)$ such that the linear span of $\Psi_\pi(\mathcal{E})\mathcal{H}$ is a dense subspace of \mathcal{K}_π and $\tilde{\pi}(x, y) = \Psi_\pi(x)^* \Psi_\pi(y)$ for all $x, y \in \mathcal{E}$. Thus Ψ_π is a non-degenerate π -map from \mathcal{E} into $\mathcal{B}(\mathcal{H}, \mathcal{K}_\pi)$.

Since (π, V, \mathcal{K}) is a dilation triple for φ and Φ_φ is a φ -map, for every $x, y \in \mathcal{E}$, we have

$$\Phi_\varphi(x)^* \Phi_\varphi(y) = \varphi(\langle x, y \rangle_{\mathcal{A}}) = V^* \pi(\langle x, y \rangle_{\mathcal{A}}) V = V^* \Psi_\pi(x)^* \Psi_\pi(y) V. \quad (1)$$

The above equation implies that for every $x_1, \dots, x_n \in \mathcal{E}$ and $h_1, \dots, h_n \in \mathcal{H}$

$$\left\| \sum_{i=1}^n \Phi_\varphi(x_i) h_i \right\|_{\mathcal{H}_\varphi} = \left\| \sum_{i=1}^n \Psi_\pi(x_i) V h_i \right\|_{\mathcal{K}_\pi}.$$

Since Φ_φ is a non-degenerate φ -map, the above equality guarantees the existence of a unique isometry $W_\varphi : \mathcal{H}_\varphi \rightarrow \mathcal{K}_\pi$ such that $W_\varphi \Phi_\varphi(x) = \Psi_\pi(x) V$ satisfies for all $x \in \mathcal{E}$. Since Φ_φ and Ψ_π are non-degenerate continuous linear maps and (π, \mathcal{K}, V) is a minimal Stinespring dilation for φ ,

$$\begin{aligned} W_\varphi(\mathcal{H}_\varphi) &= W_\varphi([\Phi_\varphi(\mathcal{E})\mathcal{H}]) = [W_\varphi \Phi_\varphi(\mathcal{E})\mathcal{H}] = [\Psi_\pi(\mathcal{E})V\mathcal{H}] \\ &= [\Psi_\pi(\mathcal{E})\pi(\mathcal{A})V\mathcal{H}] = [\Psi_\pi(\mathcal{E})[\pi(\mathcal{A})V\mathcal{H}]] = [\Psi_\pi(\mathcal{E})\mathcal{K}] = \mathcal{K}_\pi \end{aligned}$$

so W_φ is a unitary operator with the desired property. □

Proof of Theorem 2.3:

(i) Similar to the proof of Lemma 2.2, for every $x_1, \dots, x_n \in \mathcal{E}$ and $h_1, \dots, h_n \in \mathcal{H}$ we have

$$\left\| \sum_{i=1}^n \Phi(x_i)h_i \right\|_{\mathcal{H}'} = \left\| \sum_{i=1}^n \Phi_\varphi(x_i)h_i \right\|_{\mathcal{H}_\varphi}.$$

Thus there is an (onto) isometry $S_\Phi : \mathcal{H}_\varphi \rightarrow \mathcal{H}'$ such that $S_\Phi \Phi_\varphi(x)h = \Phi(x)h$ for every $x \in \mathcal{E}$ and $h \in \mathcal{H}_\varphi$. Then $\Phi(x) = S_\Phi W_\varphi^* \Psi_\pi(x)V$ for every $x \in \mathcal{E}$. Put $W := W_\varphi S_\Phi^*$, since W_φ is a unitary and S_Φ is an isometry, W is a coisometry and $\Phi(x) = W^* \Psi_\pi(x)V$ and $S_\Phi \Phi_\varphi(x) = \Phi(x)$ for every $x \in \mathcal{E}$.

For non-degenerate case, we have $[\Phi(\mathcal{E})\mathcal{H}] = \mathcal{H}'$. Hence, isometry S_Φ is onto and so it is unitary. Consequently, W is a unitary operator, too.

Conversely, each of the equations $\Phi(\cdot) = W^* \Psi_\pi(\cdot)V$ when W is a coisometry and $\Phi(\cdot) = S_\Phi \Phi_\varphi(\cdot)$ when S_Φ is an isometry, imply that Φ is a φ -map.

(ii) Let Φ be a (non-degenerate) completely semi- φ -map. For every $x_1, \dots, x_n \in \mathcal{E}$, we have

$$[\Phi(x_i)^* \Phi(x_j)]_{i,j} \leq [\varphi(\langle x_i, x_j \rangle)]_{i,j}.$$

Consequently, for every $x_1, \dots, x_n \in \mathcal{E}$ and $h_1, \dots, h_n \in \mathcal{H}$ we have

$$\left\| \sum_{i=1}^n \Phi(x_i)h_i \right\|^2 \leq \sum_{i=1}^n \sum_{j=1}^n \langle \varphi(\langle x_j, x_i \rangle)h_i, h_j \rangle = \left\| \sum_{i=1}^n \Phi_\varphi(x_i)h_i \right\|^2.$$

Thus there is a (dense range) contractive linear operator $S : \mathcal{H}_\varphi \rightarrow \mathcal{H}'$ such that $S\Phi_\varphi(x) = \Phi(x)$ for every $x \in \mathcal{E}$. Therefore $\Phi(x) = SW_\varphi^* \Psi_\pi(x)V$ for every $x \in \mathcal{E}$. Put $W := W_\varphi S^*$, since W_φ is a unitary and S is a (dense range) contractive operator, W is a (injective) contraction, and $\Phi(x) = W^* \Psi_\pi(x)V$ for every $x \in \mathcal{E}$.

Conversely, when $W : \mathcal{H}_\varphi \rightarrow \mathcal{H}'$ is a contraction, then the equation $\Phi(\cdot) = W^* \Psi(\cdot)V$ implies that Φ is a completely semi- φ -map.

(iii) It follows from (i).

(iv) For prove this, it is sufficient to set $W := S_\Psi W_\varphi S_\Phi^*$.

□

Proof of Corollary 2.4:

(i \Leftrightarrow ii): By Theorem 2.1.

(ii \Rightarrow iii): Let $\psi := \|S\|^2 \varphi$. Let $[x_{ij}] \in M_n(\mathcal{E})$. Thus

$$\begin{aligned} \Phi_n([x_{ij}])^* \Phi_n([x_{ij}]) &= [\Gamma(x_{ji})^* S^*][S\Gamma(x_{ij})] = [\Gamma(x_{ji})^*] \text{diag}(S^*, \dots, S^*) \text{diag}(S, \dots, S) [\Gamma(x_{ij})] \\ &\leq \|S\|^2 [\Gamma(x_{ji})^*][\Gamma(x_{ij})] = \|S\|^2 \varphi_n(\langle [x_{ij}], [x_{ij}] \rangle) \end{aligned}$$

and then Φ is completely semi- ψ -map.

(iii \Rightarrow iv): By part (ii) of Theorem 2.3.

(iv \Rightarrow v): See [4, Theorem 4.2]

(v \Rightarrow i): As Φ is 1 – 2 corner of some completely positive mapping on the linking C^* -algebra $\mathcal{L}(\mathcal{E})$, then Φ is a completely bounded map.

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